

Conserved mass models with stickiness and chipping

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Abstract. We introduce a chipping model in one dimensional periodic lattice with continuous mass. The model evolves following a parallel or random sequential dynamics where a fixed fraction of the mass of any site is distributed randomly among the departure site and its neighbors, and the remaining mass sticks to the site. We have calculated the steady state mass distribution of the model perturbatively for both symmetric and asymmetric mass-transfer. In most cases, the product measure turns out to be a good approximation.

Keywords: Transport processes, Stationary states, Solvable lattice models.

1. Introduction :

Most systems in nature are in non-equilibrium states [1], in a way that the accompanying fluxes of mass, energy, or spin etc. are irreversible. Unlike their equilibrium counterparts where the stationary state is characterized by the Gibbs measure, these systems usually reach different and novel stationary states depending on the dynamics of the microscopic constituents. Several non-equilibrium lattice models have been proposed recently [2, 3] to investigate the unusual steady state distributions, spatio-temporal correlations and possibility of macroscopic collective phenomena.

One of the simplest non-equilibrium model is the mass transfer model where each site of a lattice is associated with discrete masses (particles) following a dynamics that involve aggregation, fragmentation, adsorption or desorption [4, 5, 6, 7, 8]. Interestingly, these systems undergo a condensation phase transition as the density of the system is increased. Study of these model systems have generated considerable interest among physicists, as a wide variety of natural systems exhibit basic microscopic mechanism similar to that of the simple mass transport models. These includes colloidal suspensions [9], polymer gels [10, 11], river networks [12], traffic models [13], wealth distributions [14], abelian sandpile models of self organized criticality [15], zero range process [16] *etc.*

Recently, a continuous version of the mass transfer model has been proposed by Rajesh *et al.* [17] and some of its variations has been studied by others [18, 19]. Unlike the discrete mass model, where a fixed unit of mass is transferred to the neighboring sites, here a random fraction of the mass is allowed to be redistributed among the sites and their neighbors [17]. Many of these models are known to evolve into a non-equilibrium steady state that is factorized.

In this article we introduce *stickiness*, quantified by a parameter λ , to the continuous mass transfer models. At each site, $(1 - \lambda)$ fraction of the mass is chipped off (thus λ fraction of the mass sticks to the site), which is then redistributed either asymmetrically, *i.e.* among the departure site and its right neighbor, or symmetrically, *i.e.* among the two neighbors. Accordingly, the model is referred to as asymmetric or symmetric sticky chipping model respectively. We use a novel perturbation approach to calculate the steady state mass distribution of these models, for both parallel and random-sequential dynamics. Although the perturbation approach completely relies on the assumption that the steady state has a product measure, the resulting mass distributions are strikingly close to that obtained from Monte-Carlo simulations.

The article is organized as follows. The model and the perturbation method is outlined in section 2. In section 3 we study the asymmetric version of the model and obtain the steady state distribution. The symmetric version of the model, where the chipped off mass is distributed among both the neighbors, is discussed in section 4. Finally we discuss the results in section 5 and conclude.

2. The Model :

The model is defined on a one dimensional periodic lattice with sites labeled by $i = 1, 2 \dots L$. A continuous variable x_i , called mass, is associated at each site i which evolves according to the following dynamics. In each time step, $(1 - \lambda)$ fraction of the mass is chipped off from a site i (thus, λ fraction of the mass sticks to the site) and then it is distributed among the departure site i and its neighbors $i \pm 1$. In this article we study two different dynamics, (a) where the chipped off mass $(1 - \lambda)x_i$ is randomly distributed among the *two* sites, i and its right neighbor $i + 1$ and (b) where it is distributed randomly among the neighbors $(i + 1)$ and $(i - 1)$. To distribute the mass randomly among *two* sites we draw a random number r from a uniform distribution and then allow one of the sites to take r fraction of the chipped mass; the rest, *i.e.* $(1 - r)$ fraction of the chipped mass, is given to the other site. Hence forth we refer to model (a) as the *asymmetric sticky chipping model* (ASCM) as the mass is always transported to the right, whereas model (b) will be referred to as the *symmetric sticky chipping model* (SSCM). Both these models are studied with parallel and random sequential dynamics.

These models differ from the usual chipping models as for non-zero λ , there is an upper bound on what amount of the mass can be chipped off from a site, as each lattice site is constrained to retain a fixed fraction of the available mass. We must mention that for $\lambda = 0$, in model (a), r fraction of the whole mass x_i is given to the right neighbor. This model is same as asymmetric chipping model studied earlier [17], for both parallel and random sequential dynamics.

For ASCM and SSCM, we could calculate all the moments for both parallel and continuous time dynamics assuming that the steady state has a product measure, which is known to be an excellent approximation (though not exact) [19] in many models. The closed form expression for the steady state mass distribution $P(x)$ could be obtained, for $\lambda = 0$ in all cases, and for $\lambda \neq 0$ in a few specific cases.

In our effort to construct $P(x)$ from the moments, we use a novel perturbation approach. The general principle of this approach is described in the following subsection.

2.1. Perturbation approach I :

Here, we assume that all the moments of the steady mass distribution are known and then try to construct $P(x)$ perturbatively. Although $P(x)$ depends on λ , we omitted the argument for notational convenience. Let us express $P(x)$ as a power series in λ , about $\lambda = 0$ where we can obtain the exact expression of $P(x)$,

$$P(x) = P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x) + \dots = \sum_{k=0}^{\infty} \lambda^k P_k(x) \quad (1)$$

where functions $P_k(x)$ do not depend on λ . In the chipping models discussed here, the total mass of the system and hence the density $\langle x \rangle = \frac{1}{L} \sum_i x_i$ is conserved. Without any loss of generality one can fix the density to be unity. This imposes a condition on $P(x)$

$$\int_0^{\infty} dx \, x P(x) = \langle x \rangle = 1 = \int_0^{\infty} dx \, P(x) , \quad (2)$$

where the last equality stands for the normalization condition. Now, for $\lambda = 0$, $P(x) = P_0(x)$. Therefore, $P_0(x)$ satisfies *two* conditions,

$$\int_0^\infty dx P_0(x) = 1 \text{ and } \int_0^\infty dx x P_0(x) = 1. \quad (3)$$

Thus for any other $k > 0$, using (2) and (3) we have

$$\int_0^\infty dx P_k(x) = 0 = \int_0^\infty dx x P_k(x). \quad (4)$$

Since above constraints can not be satisfied by a real positive function (as $x > 0$) one can not interpret $P_k(x)$ as a probability distribution function (PDF). However, one can then obtain $P(x)$ using Eq. (1), which is assured to be a strictly, positive function, even though $P_k(x)$ are not. $P_k(x)$ can be obtained directly from knowing the moments of $P(x)$. To calculate $P_k(x)$ let us expand the moments $\langle x^n \rangle$ as a power series in λ ,

$$\langle x^n \rangle = \int_0^\infty dx x^n P(x) = \sum_{k=0}^\infty C_k^{(n)} \lambda^k. \quad (5)$$

Here $C_k^{(n)}$ are constant co-efficients (do not depend on λ), which can be determined from Eqs. (1) and (5),

$$C_k^{(n)} = \int_0^\infty dx x^n P_k(x). \quad (6)$$

As discussed above, $P_k(x)$ for any $k > 0$ is not a positive function, and thus $C_k^{(n)}$ should not be regarded as its n^{th} moment. $P_k(x)$ can be calculated from $C_k^{(n)}$ as follows.

$$P(x) = \mathcal{L}^{-1} \left[\sum_{n=0}^\infty \frac{(-s)^n}{n!} \langle x^n \rangle \right], \quad (7)$$

and from Eqs. (7), (5) and (1) it is evident that

$$P_k(x) = \mathcal{L}^{-1} \left[\sum_{n=0}^\infty \frac{(-s)^n}{n!} C_k^{(n)} \right], \quad (8)$$

where \mathcal{L}^{-1} denotes the inverse Laplace transform.

2.2. Perturbation approach II :

Here we assume that the mass distribution $P(x)$ satisfies a differential or an integral equation; the explicit form of the moments of the distribution $P(x)$ are not known. In this case, one can first use the Laplace transform \mathcal{L} on the integral equation, which usually translates into a differential or a transcendental equation in $Q(s)$ which is the Laplace transform of $P(x)$,

$$Q(s) = \mathcal{L} [P(x)] = \int_0^\infty dx e^{-sx} P(x). \quad (9)$$

We proceed further by expanding $Q(s)$ as power series in λ about $\lambda = 0$,

$$Q(s) = \sum_{k=0}^\infty \lambda^k Q_k(s), \quad (10)$$

and equate the coefficients of different powers of λ order by order. Finally one can find the distribution

$$P(x) = \mathcal{L}^{-1}[Q(s)] = \mathcal{L}^{-1}\left[\sum_{k=0}^{\infty} \lambda^k Q_k(s)\right]. \quad (11)$$

These two approaches are equivalent as it is evident from Eq. (8),

$$Q_k(s) = \mathcal{L}[P_k(x)] = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} C_k^{(n)}. \quad (12)$$

For any particular problem we will use the approach which is convenient. For ASCM, we show the equivalence explicitly, whereas for the SSCM we describe one of the approaches in details.

3. Asymmetric Sticky Chipping Model (ASCM) :

In this section we study the asymmetric version of the model, namely ASCM. Here, in each time step, λ fraction of the available mass x_i at lattice site i remains stuck to the site. From the rest $(1 - \lambda)x_i$, r_i fraction is transported to the right neighbor $i + 1$, and $(1 - r_i)$ fraction is retained at the departure site i . In other words $(1 - \lambda)r_i x_i$ is transported from site i to $i + 1$. Here r_i is a random number uniformly distributed in the interval $(0, 1)$.

First let us consider the parallel dynamics, where all the sites are updated synchronously. The dynamics can be written as

$$x_i(t+1) = \lambda x_i(t) + (1 - \lambda)(1 - r_i)x_i(t) + (1 - \lambda)r_{i-1}x_{i-1}(t) \quad (13)$$

for all $i = 1, 2, \dots, L$, where the first term represents the mass that sticks to the site i , the second term corresponds to the mass that retains at the site i after $(1 - \lambda)r_i x_i(t)$ is transported to $i + 1$. The third term results from the mass that site i receives from site $i - 1$.

A special case of the model, with $\lambda = 0$ has been studied earlier [17] for both parallel and random sequential dynamics. It was shown that only for the parallel dynamics the steady state has a product measure $P_s(\{x_i\}) = \prod_{i=1}^L P(x_i)$. For random sequential dynamics, a factorized steady state was only an excellent approximation.

In this study, we assume that $P_s(\{x_i\})$ is factorized, and try to obtain $P(x)$ using one of the perturbative approaches discussed in section 2. The results are then compared with the stationary state mass distribution obtained from the Monte-Carlo simulations of the model.

3.1. Approach I :

In order to find the steady state mass distribution, first we calculate its moments. Since, in the steady state, distribution of $x(t+1)$ is same as the distribution of $x(t)$, one may obtain $\langle x^n \rangle$ using Eq. (13) as follows,

$$\langle x^n \rangle = \sum_{k=0}^n \binom{n}{k} \left\langle \left(\lambda + (1 - \lambda)(1 - r_i) \right)^k \right\rangle \langle x^k \rangle (1 - \lambda)^{n-k} \langle r_{i-1}^{n-k} \rangle \langle x_{i-1}^{n-k} \rangle.$$

This can be simplified further,

$$\frac{\langle x^n \rangle}{(n+1)!} = \left[n - (n+1)\lambda + \lambda^{n+1} - (1-\lambda)^{n+1} \right]^{-1} \times \sum_{k=1}^{n-1} (1-\lambda^{k+1})(1-\lambda)^{n-k} \frac{\langle x^k \rangle}{(k+1)!} \frac{\langle x^{n-k} \rangle}{(n-k+1)!}. \quad (14)$$

Since, the n^{th} moment $\langle x^n \rangle$ here depends on *all* the lower moments up to $n-1$, we calculate $\langle x^n \rangle$ recursively starting from $\langle x \rangle = 1$. The first few of them are,

$$\begin{aligned} \langle x^2 \rangle &= \frac{3(\lambda+1)}{4\lambda+2} \\ \langle x^3 \rangle &= \frac{3(\lambda^2+3\lambda+2)}{2(2\lambda+1)^2} \\ \langle x^4 \rangle &= \frac{15(\lambda+1)^2(\lambda^2-\lambda+3)}{2(2\lambda+1)^2(2\lambda^3-\lambda^2+6\lambda+3)} \\ &\vdots \end{aligned} \quad (15)$$

The moments $\langle x^n \rangle$ as a function of λ becomes messy with increase of n , and obtaining a general expression becomes practically impossible. Thus it would be useful to obtain $P(x)$ perturbatively which gives *all* the moments correctly up to some order in λ .

To proceed with the perturbative approach, first let us express $\langle x^n \rangle$ as a power series in λ as done in Eq. (5),

$$\langle x^n \rangle = \sum_{k=0}^{\infty} C_k^{(n)} \lambda^k = (n+1)! \sum_{k=0}^{\infty} A_k^{(n)} \lambda^k, \quad (16)$$

where we have used, for convenience

$$C_k^{(n)} = (n+1)! A_k^{(n)}. \quad (17)$$

From the Taylor's series expansion of Eq. (14) in λ about $\lambda = 0$, we equate the coefficients of different powers of λ and get a set of recursive equations for $A_k^{(n)}$. It is evident that $A_k^{(n)}$ depends on $A_{k'}^{(n')}$ with $0 \leq k' < k$ and $0 \leq n' < n$. These equations can be solved recursively as they are supplemented by the boundary conditions

$$A_0^{(0)} = C_0^{(0)} = 1, \quad A_0^{(1)} = \frac{1}{2} C_0^{(1)} = \frac{1}{2}, \quad (18)$$

where $C_0^{(0)} = 1 = C_0^{(1)}$ corresponds to the normalization of the steady distribution function and the conservation of mass respectively (see Eqs. (3) and (6)). We can then obtain

$$P_k(x) = \mathcal{L}^{-1} \left[\sum_{n=0}^{\infty} (-s)^n (n+1) A_k^{(n)} \right], \quad (19)$$

using Eq. (8) and finally calculate $P(x)$ using Eq. (1).

0^{th} order perturbation in λ :

In the perturbative approach the 0^{th} order corresponds to the system with no stickiness. In this case, Eqs. (14) and (16) results in

$$(n-1)A_0^{(n)} = \sum_{k=1}^{n-1} A_0^{(k)} A_0^{(n-k)} \quad \forall n \geq 2, \quad (20)$$

which needed to be solved using the boundary condition $A_0^{(1)} = \frac{1}{2}$ (which is equivalent to fixing the density of the system to unity). This results in

$$A_0^{(n)} = \frac{1}{2^n} \quad \forall n \geq 0. \quad (21)$$

Further using Eq. (19) we get the steady state distribution function as

$$P_0(x) = 4xe^{-2x}. \quad (22)$$

The mass distribution function Eq. (22) is identical to the steady state distribution obtained earlier for the model with no stickiness (*i.e.* for $\lambda = 0$) where the steady state is known to have product measure [17].

In Fig. 1 we show $P(x)$ obtained from Monte-Carlo simulation of a system of size $L = 100$ for different λ . Fig. 1(b) compares $P(x)$ for $\lambda = 0$ with Eq. (22) (solid line).

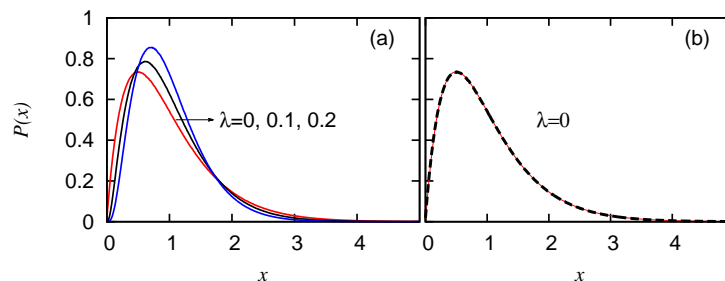


Figure 1. (a) Steady state mass distribution $P(x)$ of ASCM with parallel dynamics obtained from Monte-Carlo simulations of system size $L = 100$ for $\lambda = 0, 0.1$, and 0.2 . (b) $P(x)$ for $\lambda = 0$ (dashed line) is compared with Eq. (22).

In presence of stickiness, we will show that $P(x)$ can be obtained with reasonable accuracy when perturbative correction terms are added to $P_0(x)$ order by order.

1^{st} order perturbation in λ :

Next we proceed to calculate the distribution function up to the first order in λ . Using Eq. (16) and comparing the coefficients of λ in Eq. (14) we obtain

$$(n-1)A_1^{(n)} = \sum_{k=1}^{n-1} [2A_0^{(k)} A_1^{(n-k)} - (n-k)A_0^{(k)} A_0^{(n-k)}]$$

which are needed to be solved using boundary conditions $A_1^0 = 0, A_1^1 = 0$ (obtained from Eqs. (4), (6) and (17)). Solving the above equation, we obtain

$$A_1^{(n)} = 2^{-n}n(1 - \gamma - \psi^{(0)}(n+1)) \quad (23)$$

where γ is the *Euler constant*,

$$\gamma = 0.57721... \quad (24)$$

and $\psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the *Digamma* function [20, 22]. Using Eq. (19) we get

$$\begin{aligned} P_1(x) &= \mathcal{L}^{-1} \left[\sum_{n=0}^{\infty} \frac{(-s)^n}{n!} C_1^{(n)} \right] = \mathcal{L}^{-1} \left[\sum_{n=0}^{\infty} (-s)^n (n+1) A_1^{(n)} \right] \\ &= 8e^{-2x} \left[-x^2 \{g(x) - 1\} + xg(x) - \frac{1}{4} \right] \end{aligned} \quad (25)$$

$$\text{where } g(x) = \ln(2x) + \gamma \quad (26)$$

and γ is given in Eq. (24).

To check how good is the 1st order correction, we compare $P(x) = P_0(x) + \lambda P_1(x)$ with the steady state distribution obtained from the Monte-Carlo simulations of ASCM on a lattice of size $L = 100$, for two different $\lambda = 0.1, 0.2$ (see Fig. 2(a) and (b)).

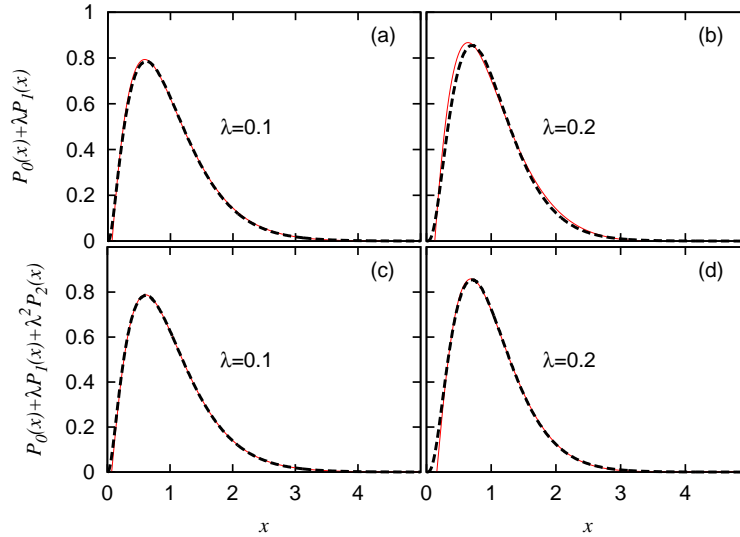


Figure 2. (a) and (b): The steady state mass distribution obtained from Monte-Carlo simulations (dashed line) of ASCM on a lattice of size $L = 100$ with parallel dynamics for $\lambda = 0.1, 0.2$ are compared with the 1st order perturbation results (solid lines). (c) and (d): Comparison of the same by the taking 2nd order correction.

Clearly for $\lambda = 0.1$, the perturbative result is in good agreement with $P(x)$ obtained from the Monte-Carlo simulations, despite the fact that we have assumed product measure for the steady state. For $\lambda = 0.2$, $P_0(x) + \lambda P_1(x)$ shows a small departure, which can possibly be substantiated by adding the 2nd order correction.

2nd order perturbation in λ :

To calculate the 2nd order correction term in λ we proceed in a similar way; first we obtain the recursion relation for $A_2^{(n)}$,

$$\begin{aligned} (n-1)A_2^{(n)} = & \sum_{k=1}^{n-1} \left[2A_0^{(k)} A_2^{(n-k)} + A_1^{(k)} A_1^{(n-k)} - (n-k)A_0^{(k)} A_1^{(n-k)} \right. \\ & \left. - (n-k)A_0^{(n-k)} A_1^{(k)} + \frac{(n-k)(n-k+1)}{2} A_0^{(k)} A_0^{(n-k)} \right] \\ & - A_0^{(1)} A_0^{(n-1)} + \frac{n(n+1)}{2} A_0^{(n)}. \end{aligned} \quad (27)$$

Since $A_0^{(n)}$ and $A_1^{(n)}$ are known function of n (see Eqs. (21) and (23)), the above equation can be solved by using

$$V_2(s) = \sum_{n=0}^{\infty} (-s)^n A_2^{(n)}, \quad (28)$$

which is the generating function for $A_2^{(n)}$. Substituting for $A_0^{(n)}$ and $A_1^{(n)}$ in Eq. (27), multiplying both sides of the equation by $(-s)^n$ and taking the sum over n we obtain a differential equation for $V_2(s)$;

$$\begin{aligned} & s(s+2)^4 V_2'(s) + (s-2)(s+2)^3 V_2(s) \\ & - s^2 \left[s^2 + 2s + 4 \ln^2 \left(\frac{s+2}{2} \right) + 2(s-4) \ln \left(\frac{s+2}{2} \right) + 8 \right] = 0. \end{aligned} \quad (29)$$

This equation can be solved using the boundary condition $V_2'(0) = 0$ (obtained from the condition $\langle x \rangle = 1$), resulting

$$V_2(s) = \frac{s}{(2+s)^3} \left[s^2 + 4s - 2s \ln \left(1 + \frac{s}{2} \right) + (s-2) \ln^2 \left(1 + \frac{s}{2} \right) \right]. \quad (30)$$

Now $A_2^{(n)}$ is the coefficient of $(-s)^n$ in the power series expansion of $V_2(s)$,

$$\begin{aligned} A_2^{(n)} = & \frac{(-1)^n}{(n+1)^{2n+1}} \left[-2 - n + n^2(4 + 2\gamma) + 3n^3 + 2n^2 \{-2 + \gamma + (-1 + \gamma)n\} H_n^{(1)} \right. \\ & \left. - n^2(n+1) \{ \gamma^2 + H_{n+1}^{(2)} - \{ \psi^{(0)}(2+n) \}^2 \} \right], \end{aligned}$$

where $H_n^{(r)}$ are Harmonic numbers [20]. Thus,

$$\begin{aligned} P_2(x) = & \mathcal{L}^{-1} \left[\sum_{n=0}^{\infty} (-s)^n (n+1) A_2^{(n)} \right] \\ = & 2e^{-2x} \left[4x^3 \left(g^2(x) - 2g(x) - \frac{\pi^2}{6} + 3 \right) - 2x^2 \left(5g^2(x) - 4g(x) - 5\frac{\pi^2}{6} + 8 \right) \right. \\ & \left. + 2x \left(2g^2(x) + 3g(x) - 2\frac{\pi^2}{6} \right) - \{g(x) + 1\} \right] + 2\delta(x) \end{aligned} \quad (31)$$

where $g(x) = \ln(2x) + \gamma$. Note that in the above expression the term $2\delta(x)$ is essential to assure that $\int_0^\infty dx P_2(x) = 0$. In Fig. 2(c) and (d) we compare $P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x)$

with $P(x)$ obtained from the Monte-Carlo simulations of the dynamics (13) on a ring of size $L = 100$ for *two* different $\lambda = 0.1, 0.2$. It is evident that for large λ the 2^{nd} order correction provides a substantial improvement over 1^{st} order correction.

3.2. Approach II :

The steady state distribution can also be obtained from the second approach discussed in the subsection 2.2. To obtain an equation that $P(x)$ must satisfy in steady state, we proceed by writing a master equation, for the dynamics (13),

$$\frac{d}{dt}P(x'_i) = -P(x'_i) + \int_0^\infty dx_{i-1} \int_0^\infty dx_i \int_0^1 dr_i \int_0^1 dr_{i-1} f(r_i) f(r_{i-1}) P(x_i, x_{i-1}) \\ \times \delta(x'_i - \{\lambda + (1 - \lambda)(1 - r_i)\}x_i - (1 - \lambda)r_{i-1}x_{i-1}),$$

where $P(x_i, x_{i-1})$ is the two point joint distribution function. In the steady state,

$$P(x'_i) = \int_0^\infty dx_{i-1} \int_0^\infty dx_i \int_0^1 dr_i \int_0^1 dr_{i-1} f(r_i) f(r_{i-1}) P(x_i, x_{i-1}) \\ \times \delta(x'_i - \{\lambda + (1 - \lambda)(1 - r_i)\}x_i - (1 - \lambda)r_{i-1}x_{i-1}).$$

For $\lambda = 0$, it has been already indicated in [17] that both the parallel and random sequential dynamics of the model can be treated by taking the probability distribution function of r_i as

$$f(r_i) = p + (1 - p) \delta(r_i) \quad (32)$$

where $0 \leq p < 1$ is a fixed number. Effectively the random variable r_i is chosen to be 0 with probability $(1 - p)$ or otherwise it is chosen uniformly from the interval $(0, 1)$. For $p = 1$ the dynamics corresponds to the parallel update and in the $p \rightarrow 0$ limit the dynamics is equivalent to the random sequential update.

Let us assume that the steady state mass distribution of the model has a product measure, *i.e.* there is no correlation among the sites. Thus

$$P(x'_i) = \int_0^\infty dx_{i-1} \int_0^\infty dx_i \int_0^1 dr_i \int_0^1 dr_{i-1} f(r_i) f(r_{i-1}) P(x_i) P(x_{i-1}) \\ \times \delta(x'_i - \{\lambda + (1 - \lambda)(1 - r_i)\}x_i - (1 - \lambda)r_{i-1}x_{i-1}). \quad (33)$$

Now, taking the Laplace transform on both sides of the above equation we obtain

$$Q(s) \left[-p(1 - p)V(s(1 - \lambda)) - (1 - p)^2 + 1 \right] \\ = \left[p^2 V(s(1 - \lambda)) + p(1 - p) \right] \int_0^1 dr Q(sr + \lambda s(1 - r)) \quad (34)$$

where $Q(s)$ is the Laplace transform of $P(x)$, and $V(s)$ is defined by

$$V(s) = \int_0^1 dr Q(sr), \quad \text{or} \quad Q(s) = \frac{d}{ds} \{sV(s)\}. \quad (35)$$

In fact $Q(s)$ can be obtained from Eq. (34), and one can rewrite it *only* in terms of the function V as

$$\frac{d}{ds} \{sV(s)\} \left[-p(1 - p)V(s(1 - \lambda)) - (1 - p)^2 + 1 \right] \\ = \left[p^2 V(s(1 - \lambda)) + p(1 - p) \right] \int_0^1 dr \sum_{m=0}^{\infty} \frac{\{\lambda s(1 - r)\}^m}{m!} \left[\frac{d^{m+1}}{ds^{m+1}} \{sV(s)\} \right]_{s=sr} \quad (36)$$

where we have expanded $Q(sr + \lambda s(1 - r))$ in Taylor series about $\lambda = 0$, and used Eq. (35). Since this differential equation incorporates infinite number of terms, it is not easy to find a general solution. We again use the perturbative approach as follows. If the argument of the function V depends on λ , we first get a Taylor series expansion of V in λ about $\lambda = 0$,

$$V(s(1 - \lambda)) = \sum_{m=0}^{\infty} \frac{(-\lambda s)^m}{m!} \frac{d^m}{ds^m} \{V(s)\}. \quad (37)$$

Each of the $V(s)$ and their derivatives are then expanded using

$$V(s) = \sum_{k=0}^{\infty} \lambda^k V_k(s). \quad (38)$$

Now using Eqs. (37) and (38) in Eq. (36) and collecting the coefficients of different powers of λ , order by order, one can obtain a set of differential equations in terms of $V_k(s)$.

Finally we proceed to calculate $V_k(s)$ for specific dynamics, *i.e.* parallel ($p = 1$) and random sequential ($p \rightarrow 0$) by using the boundary conditions,

$$\begin{aligned} k = 0 & : & V_0(0) = 1 & , & V'_0(0) = -\frac{1}{2} \\ k \neq 0 & : & V_k(0) = 0 & , & V'_k(0) = 0 \end{aligned} \quad (39)$$

which comes from the generic conditions (3) and (4), correspond to the normalization of the probability distribution function and that the mass density is unity.

Some comments are in order. Note that from Eq. (35) we have

$$V_k(s) = \int_0^1 dr Q_k(sr) \quad \text{or} \quad Q_k(s) = \frac{d}{ds} \{s V_k(s)\}, \quad (40)$$

where $Q(s) = \sum_{k=0}^{\infty} \lambda^k Q_k(s)$. Again, since $Q_k(s) = \sum_{n=0}^{\infty} (-s)^n (n+1) A_k^{(n)}$,

$$V_k(s) = \sum_{n=0}^{\infty} (-s)^n A_k^{(n)}. \quad (41)$$

Thus $V_k(s)$ is simply a generalization of $V_2(s)$ defined in Eq. (28). In other words $V_k(s)$ is the generating function of $A_k^{(n)}$.

Now, let us consider the specific case $p = 1$ corresponding to parallel dynamics and proceed from Eq. (36) to obtain the coefficients of different powers of λ , order by order.

0th order perturbation in λ :

The terms independent of λ in Eq. (36) gives

$$sV'_0(s) - V_0(s)^2 + V_0(s) = 0$$

which is solved using the boundary condition (39):

$$V_0(s) = \frac{2}{s+2}. \quad (42)$$

This gives

$$P_0(x) = \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_0(s)\} \right] = 4xe^{-2x}.$$

$P_0(x)$ is identical to the exact mass distribution function obtained earlier [17] for $\lambda = 0$.

1st order perturbation in λ :

Equating coefficient of λ in Eq. (36), we get

$$s\{V_0(s) - 1\}V_0'(s) + sV_1'(s) + \{1 - 2V_0(s)\}V_1(s) = 0$$

where $V_0(s)$ is given in Eq. (42). This can be solved using boundary condition (39).

$$V_1(s) = -\frac{2s \ln\left(\frac{s+2}{2}\right)}{(s+2)^2}, \quad (43)$$

which results in

$$\begin{aligned} P_1(x) &= \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_1(s)\} \right] \\ &= 8e^{-2x} \left[-x^2 \{g(x) - 1\} + xg(x) - \frac{1}{4} \right] \end{aligned} \quad (44)$$

where $g(x) = \ln(2x) + \gamma$ and γ is *Euler constant*. Thus we obtain the same 1st order correction as in Eq. (25) obtained using approach 3.1.

2nd order perturbation in λ :

The coefficients of λ^2 in Eq. (36), gives

$$\begin{aligned} &-\frac{1}{2}V_0(s) [s\{sV_0''(s) - 2V_1'(s) + 1\} + 4V_2(s)] + sV_1(s)V_0'(s) - sV_0'(s) \\ &-sV_1'(s) - V_1(s)^2 + sV_2'(s) + V_2(s) = 0 \end{aligned}$$

where $V_0(s)$ and $V_1(s)$ are given by Eq. (42) and (43) respectively. One can solve this differential equation subjected to the boundary condition (39),

$$V_2(s) = \frac{s}{(2+s)^3} \left[s^2 + 4s - 2s \ln\left(1 + \frac{s}{2}\right) + (s-2) \ln^2\left(1 + \frac{s}{2}\right) \right].$$

Thereafter $P_2(x)$ is obtained as

$$\begin{aligned} P_2(x) &= \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_2(s)\} \right] \\ &= 2e^{-2x} \left[4x^3 \left(g^2(x) - 2g(x) - \frac{\pi^2}{6} + 3 \right) - 2x^2 \left(5g^2(x) - 4g(x) - 5\frac{\pi^2}{6} + 8 \right) \right. \\ &\quad \left. + 2x \left(2g^2(x) + 3g(x) - 2\frac{\pi^2}{6} \right) - \{g(x) + 1\} \right] + 2\delta(x) \end{aligned} \quad (45)$$

where $g(x) = \ln(2x) + \gamma$. This expression for $P_2(x)$ is same as in Eq. (31).

The mass distribution obtained using perturbative approach upto 2^{nd} order is compared to that of the Monte-Carlo simulations in Fig. 2.

In the following section we study the model ASCM with random sequential dynamics.

3.3. Random sequential dynamics :

In this subsection we study the *asymmetric sticky chipping model (ASCM)* with random sequential update where at each time step a site i is chosen randomly, and from this site, the mass $(1 - \lambda)r_i x_i$ is transported to the right neighbor $i + 1$. Rest of the mass $\{\lambda + (1 - \lambda)(1 - r_i)\}x_i$ remain at the departure site i . As usual, r_i is a random number distributed uniformly in the interval $(0, 1)$.

Like the parallel dynamics, here too one can obtain all the moments exactly in terms of λ assuming that the steady state has a product measure. Then we use the perturbative approach similar to the previous case of parallel dynamics. Clearly the moments $\langle x^n \rangle$ are given by

$$\langle x_i^n \rangle = \frac{1}{2} \langle \{\lambda + (1 - \lambda)(1 - r_i)\}^n x_i^n \rangle + \frac{1}{2} \langle \{x_i + (1 - \lambda)r_{i-1}x_{i-1}\}^n \rangle, \quad (46)$$

for all $i = 1, 2, \dots, L$, where, the 1^{st} term represents that on the average half of the time λx_i amount of the mass sticks to the site i and the rest is chipped off. Of this chipped off mass $(1 - \lambda)x_i$, the amount $(1 - \lambda)r_i x_i$ is transported to site $i + 1$ and the rest is retained at the departure site i . Thus half of the time site i retains $\{\lambda + (1 - \lambda)(1 - r_i)\}x_i$ amount of mass and in the other half i receives $(1 - \lambda)r_{i-1}x_{i-1}$ mass from site $i - 1$.

3.3.1. Approach I :

We begin with Eq. (46) and $\langle x^n \rangle$ in terms of lower moments $\langle x^m \rangle$ with $m < n$:

$$\begin{aligned} \frac{\langle x^n \rangle}{(n+1)!} &= \left[n - (n+1)\lambda + \lambda^{n+1} - (1-\lambda)^{n+1} \right]^{-1} \\ &\times \sum_{k=1}^{n-1} (n-k+1)(1-\lambda)^{k+1} \frac{\langle x^k \rangle}{(k+1)!} \frac{\langle x^{n-k} \rangle}{(n-k+1)!}. \end{aligned} \quad (47)$$

After expanding the moments $\langle x^n \rangle$ as power series in λ about $\lambda = 0$ as $\langle x^n \rangle = \sum_{k=0}^{\infty} \lambda^k C_k^{(n)}$ where the coefficients $C_k^{(n)}$ are independent of λ . For convenience we choose $C_k^{(n)} = (n+1)! A_k^{(n)}$. Using these in Eq. (47) and then equating the coefficients of different powers of λ , we obtain a set of recursive relations which are needed to be satisfied by $A_k^{(n)}$ for different orders. However obtaining the solution for $A_k^{(n)}$ from those recursive relations, is usually difficult. Instead one can use $V_k(s)$, the generating functions for $A_k^{(n)}$ (see Eq. (41)) and then calculate $P_k(x)$ by using Eqs. (12) and (40), $P_k(x) = \mathcal{L}^{-1}[Q_k(s)] = \mathcal{L}^{-1}[\frac{d}{ds}\{sV_k(s)\}]$.

0th order perturbation in λ :

Coefficient of terms independent of λ in Eq. (47) gives

$$(n-1)A_0^{(n)} = \sum_{k=1}^{n-1} (n-k+1)A_0^{(k)}A_0^{(n-k)} \quad (48)$$

Multiplying both sides of Eq. (48) by $(-s)^n$ and taking sum over n we obtain an equation in terms of the generating function $V_0(s)$ define in (41),

$$sV_0(s)V_0'(s) - 2sV_0'(s) + V_0^2(s) - V_0(s) = 0.$$

Since $V_0(s)$ satisfies the boundary condition (39), we get the solution,

$$V_0(s) = \frac{\sqrt{2s+1} - 1}{s} \quad (49)$$

which gives

$$P_0(x) = \mathcal{L}^{-1}\left[\frac{d}{ds}\{sV_0(s)\}\right] = \frac{e^{-x/2}}{\sqrt{2\pi}\sqrt{x}}. \quad (50)$$

Note that $P_0(x)$ is same as the mass distribution of the ASCM with $\lambda = 0$, obtained earlier [17] for random sequential dynamics. The steady state distribution function for ASCM with random sequential dynamics obtained from Monte-Carlo simulation of system size $L = 100$ for $\lambda = 0, 0.1$ and 0.2 , is shown in Fig. 3(a). In Fig. 3(b) we compare $P_0(x)$ with $P(x)$ obtained from Monte-Carlo simulations.

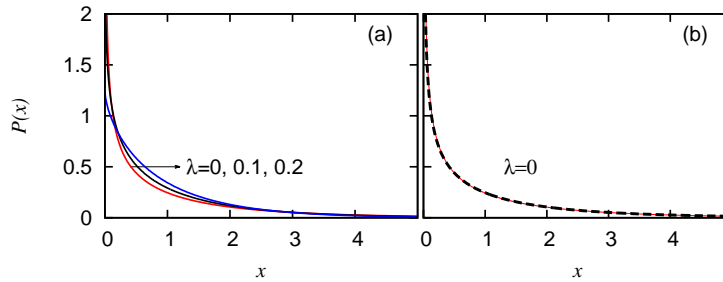


Figure 3. (a) The steady state distribution $P(x)$ of ASCM with random sequential update obtained from Monte-Carlo simulations of system size $L = 100$ for $\lambda = 0, 0.1$ and 0.2 . (b) $P(x)$ for $\lambda = 0$ is compared with Eq. (50).

1st order perturbation in λ :

Collecting the coefficient of λ in Eq. (47) one can get

$$(n-1)A_1^{(n)} = \sum_{k=1}^{n-1} \left[-(n-k+1)(k+1)A_0^{(k)}A_0^{(n-k)} + (n-k+1) \right. \\ \left. (A_0^{(k)}A_1^{(n-k)} + A_1^{(k)}A_0^{(n-k)}) \right].$$

This recursion relation can be converted to a differential equation in terms of the generating functions $V_0(s)$ and $V_1(s)$.

$$-sV_0(s)V_1'(s) + 2sV_1'(s) - sV_0'(s)V_1(s) - 2V_0(s)V_1(s) + V_1(s) + sV_0(s)V_0'(s) + s^2V_0'^2(s) - V_0(s) + 1 = 0$$

where $V_0(s)$ is given in Eq. (49). Using the boundary condition (39) the solution is obtained as

$$V_1(s) = -\frac{2s(\check{s} - 3) + 4(\check{s} - 1) + 2(s - \check{s} + 1)\ln(\check{s})}{2s\check{s}} \quad (51)$$

where, $\check{s} = \sqrt{1 + 2s}$. Thus we get

$$\begin{aligned} P_1(x) &= \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_1(s)\} \right] \\ &= \frac{1}{8} e^{-x/2} \left[\frac{\sqrt{2}}{\sqrt{\pi}} \left\{ -x^{1/2} \{g(x) + 2\} + x^{-1/2} \{g(x) + 4\} \right\} + 4 \right] - 2\delta(x) \end{aligned} \quad (52)$$

where $g(x) = \ln(2x) + \gamma$. In the above expression the term $-2\delta(x)$ is needed to ensure the condition $\int_0^\infty dx P_1(x) = 0$.

The perturbative results up to 1st order correction are then compared with $P(x)$ obtained from the Monte-Carlo simulations of the model with different λ .

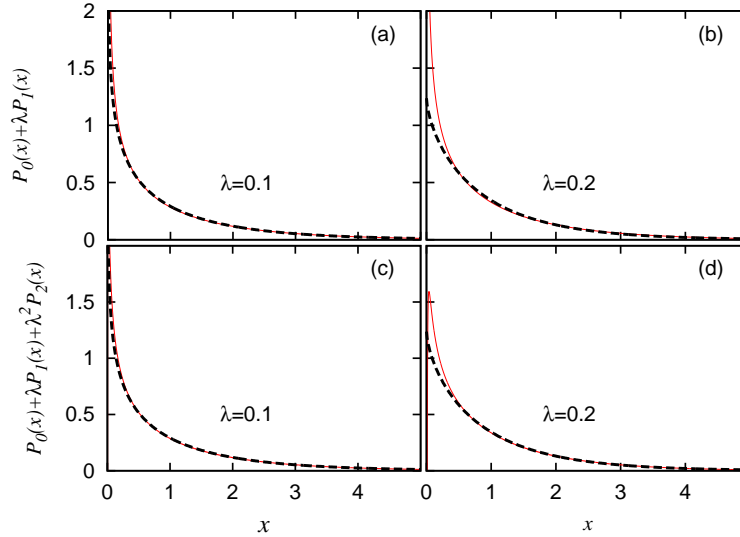


Figure 4. $P(x)$ obtained from Monte-Carlo simulations of ASCM with random sequential dynamics on a lattice of size $L = 100$ are compared with the perturbative results for $\lambda = 0.1$ and 0.2 . In the upper ((a) and (b)) and lower ((c) and (d)) panel 1st and 2nd order correction are being taken into account respectively.

Fig. 4 shows that for $\lambda = 0.1$, the perturbatively obtained result is very close to that obtained from Monte-Carlo simulations whereas for $\lambda = 0.2$ there is a fair amount of discrepancy which may be improved by adding higher order corrections.

2^{nd} order perturbation in λ :

From Eq. (47) collecting the coefficient of λ^2 we have the equation

$$\begin{aligned} (n-1)A_2^{(n)} = & \sum_{k=1}^{n-1} (n-k+1) \left[\frac{k(k+1)}{2} A_0^{(k)} A_0^{(n-k)} + A_0^{(k)} A_2^{(n-k)} \right. \\ & + A_2^{(k)} A_0^{(n-k)} + A_1^{(k)} A_1^{(n-k)} - (k+1)(A_0^{(k)} A_1^{(n-k)} \\ & \left. + A_1^{(k)} A_0^{(n-k)}) \right] + \frac{n(n+1)}{2} A_0^{(n)}. \end{aligned} \quad (53)$$

Using the generating functions $V_0(s)$, $V_1(s)$ and $V_2(s)$ from Eq. (41), we get

$$-\frac{1}{2}s^3V_0'(s)V_0''(s) - \frac{1}{2}s^2V_0(s)V_0''(s) + 2s^2V_0'(s)V_1'(s) + sV_1(s)V_0'(s) - sV_2(s)V_0'(s) + sV_0(s)V_1'(s) - sV_0(s)V_2'(s) - 2V_0(s)V_2(s) - V_0(s) - sV_1(s)V_1'(s) - V_1(s)^2 - V_1(s) + 2sV_2'(s) + V_2(s) - \frac{s}{2} + 1 = 0.$$

Since $V_0(s)$ and $V_1(s)$ are already known from Eq. (49) and (51) respectively, the solution of the above equation can be obtained using the boundary conditions (39):

$$\begin{aligned} V_2(s) = & \frac{1-\check{s}}{24\check{s}^4(1+\check{s})} \left[-16(\check{s}-1) - 4s\{s(4s+9\check{s}+6) + 26\check{s}-6\} \right. \\ & \left. + 4\{3-3s(\check{s}-2)\} \ln^2(\check{s}) + 24\{s(4\check{s}-2) + \check{s}-1\} \ln \check{s} \right], \end{aligned}$$

which results in

$$\begin{aligned} P_2(x) = & \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_2(s)\} \right] \\ = & \frac{1}{2}e^{-\frac{x}{2}} \left[-\frac{5}{8\sqrt{2\pi}}x^{-3/2} + \frac{1}{16\sqrt{2\pi}} \left(x^{3/2} \{ \{g(x)+2\}^2 + 6 - \frac{\pi^2}{2} \} \right. \right. \\ & - 4x^{1/2} \{ \{g(x)+3\}^2 - \frac{11}{6} - \frac{\pi^2}{2} \} + x^{-1/2} \{ \{g(x)+6\}^2 - \frac{118}{3} - \frac{\pi^2}{2} \} \Big) \\ & \left. - \frac{1}{4}x \{g(x) - 2\ln(2) + 3\} + \frac{1}{2} \{g(x) - 2\ln(2) + 5\} \right] - \frac{4\delta(x)}{3} + \frac{2}{3}\delta'(x) \end{aligned} \quad (54)$$

where $g(x) = \ln(2x) + \gamma$. In the above expression the last *two* terms ensure normalization of probability and conservation of mass respectively (see Eq. (4)).

In Fig. 4 (c) and (d) we compare $P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x)$ with the mass distribution obtained from Monte-Carlo simulations of a system of size $L = 100$.

3.3.2. Approach II ($\lim_{p \rightarrow 0}$) :

As discussed in section 3.2, ASCM with random sequential dynamics can be considered as a special case ($p \rightarrow 0$ limit) of parallel update scheme where the distribution of the random number is given by Eq. (32). In this limit terms $\mathcal{O}(p^2)$ can

be neglected in Eq. (36), which gives

$$\begin{aligned} & \frac{d}{ds} \{sV(s)\} [2 - V(s(1-\lambda))] \\ &= \int_0^1 dr \sum_{m=0}^{\infty} \frac{\{\lambda s(1-r)\}^m}{m!} \left[\frac{d^{m+1}}{ds^{m+1}} \{sV(s)\} \right]_{s=sr} \end{aligned} \quad (55)$$

Solving this equation perturbatively we obtain the same $P_0(x)$, $P_1(x)$ and $P_2(x)$ which turn out to be same as Eqs. (50), (52) and (54) respectively.

3.3.3. An analytically solvable case, $\lambda = 1/2$:

It is interesting to note that the asymmetric sticky chipping model (ASCM) with continuous time update, can be solved analytically at $\lambda = 1/2$. In this subsection we study this case in details. The dynamics is as follows. At each time step *one* site i with mass x_i is chosen randomly. Half of the amount ($\frac{x_i}{2}$) sticks to the site i . Of the rest half, $\frac{x_i}{2}r_i$ is transported to the right neighbor $i+1$ and rest is retained at site i . In other words the mass $\frac{x_i}{2}r_i$ is transported to site $(i+1)$ from site i and the mass $\frac{(2-r_i)x_i}{2}$ remain at the departure site i . In this case we obtain $P(x)$ as follows. From Eq. (46) we get the equation

$$\langle x_i^n \rangle = \frac{1}{2} \left\langle \left\{ \frac{(2-r_i)x_i}{2} \right\}^n \right\rangle + \frac{1}{2} \left\langle \left\{ x_i + \frac{r_{i-1}x_{i-1}}{2} \right\}^n \right\rangle$$

Simplifying we get an equation that expresses the n^{th} term in terms of lower moments up to $n-1$:

$$(n-1) \frac{\langle x^n \rangle}{(n+1)!} = \sum_{k=1}^{n-1} \frac{n-k+1}{2^k} \frac{\langle x^k \rangle}{(k+1)!} \frac{\langle x^{n-k} \rangle}{(n-k+1)!}. \quad (56)$$

With the boundary condition $\langle x \rangle = 1$ the solution is obtained as

$$\langle x^n \rangle = \frac{(n+1)!}{2^n} \quad (57)$$

and hence

$$P(x) = \mathcal{L}^{-1} \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \langle x^n \rangle = 4xe^{-2x}. \quad (58)$$

Thus in this case we obtain the closed form expression for the steady state mass distribution function. To check whether the result is consistent we compare our result with Monte-Carlo simulations of a system of size $L = 100$.

Fig. (5(a)) shows that Eq. (58) agrees quite well with the distribution $P(x)$ obtained from Monte-Carlo simulations. We have checked that the small discrepancy near the peak (shown in the inset of Fig. (5)) does not disappear when the system size is increased. This small discrepancy observed indicates that possibly the steady state in this case is not *actually* factorized.

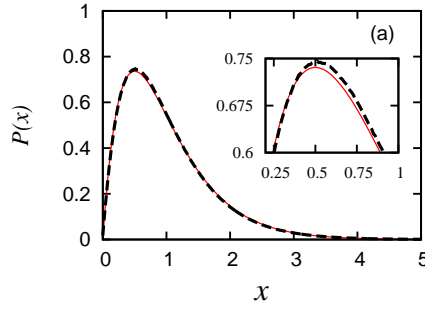


Figure 5. (a) The steady state distribution function $P(x)$ (dashed line) of ASCM with random sequential dynamics for $\lambda = 1/2$ obtained from Monte-Carlo simulations of system size $L = 100$, is compared with Eq. (58) (solid line).

4. Symmetric Sticky Chipping Model (SSCM) :

In this section we study the symmetric version of the model, namely SSCM. Here, the dynamics is as follows. In each time step at site i with mass x_i , λ fraction of the mass sticks to the site and $(1 - \lambda)x_i$ is chipped off and distributed randomly among both the neighbors of i *i.e.* $(i \pm 1)$ in a way that one of those receives $r_i(1 - \lambda)x_i$ and the other receives the rest, $(1 - r_i)(1 - \lambda)x_i$, where r_i is a random number uniformly distributed in the interval $(0,1)$. We will use *two* different update schemes - parallel update and random sequential update.

4.1. Parallel Update :

In this subsection we study the model where all the sites are updated parallelly (synchronously) using the dynamics mentioned above. Mathematically, the dynamics can be explicitly written as

$$x_i(t+1) = \lambda x_i(t) + (1 - \lambda)[r_{i-1}x_{i-1}(t) + r_{i+1}x_{i+1}(t)] \quad (59)$$

where, the 1st term on the r.h.s. represents the mass that sticks to the site i during the update, 2nd and 3rd term correspond to the mass which site i receives from its neighbors $i \mp 1$ respectively.

The steady state mass distribution function can be obtained in the same approach as shown in case of ASCM. Here we follow the approach as shown in section 3.2. Converting the dynamical equation for the system (Eq. (59)) into an integral equation of $P(x)$ (as shown in case of ASCM) and taking the Laplace transform of both sides of that equation, we get

$$Q(s) = Q(\lambda s)V^2((1 - \lambda)s)$$

which can be expressed in terms of the function V *only*, using Eq. (35),

$$\frac{d}{ds}\{sV(s)\} = \left[\frac{d}{ds}\{V(s)\} \right]_{s=\lambda s} V^2((1 - \lambda)s). \quad (60)$$

Note that $V(s)$ is also a function of λ . Here we have dropped the argument λ in $V(s)$ for notational convenience. We obtain the solution of this equation using the perturbative approach by taking the Taylor series expansion of both sides of Eq. (60), and equating the coefficients of different powers of λ order by order.

0th order perturbation in λ :

From Eq. (60) we equate the coefficient of the terms independent of λ , which gives

$$sV_0'(s) - V_0(s)^2 + V_0(s) = 0$$

This can be solved using the boundary conditions (39):

$$V_0(s) = \frac{2}{s+2}.$$

This results in

$$P_0(x) = 4xe^{-2x}.$$

This expression is identical to that obtained in case of *asymmetric sticky chipping model* with parallel update at $\lambda = 0$ and the expression obtained in case of *asymmetric sticky chipping model* with random sequential update at $\lambda = 1/2$. The steady state distribution function $P(x)$ obtained from Monte-Carlo simulations of SSCM with parallel dynamics for $\lambda = 0, 0.1$ and 0.2 with system size $L = 100$ is shown in Fig. 6(a). Fig. 6(b) corresponds to the comparison of $P_0(x)$ with Monte-Carlo simulations.

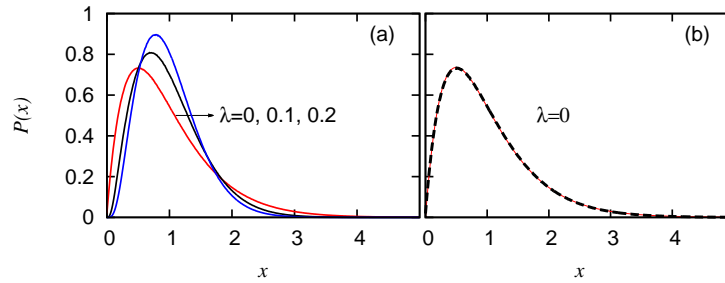


Figure 6. (a) shows $P(x)$ obtained from Monte-Carlo simulations of SSCM with parallel dynamics for system size $L = 100$ for $\lambda = 0, 0.1$ and 0.2 . (b) compares $P_0(x)$ with $P(x)$ obtained from Monte-Carlo simulations.

1st order perturbation in λ :

In Eq. (60) collecting the coefficients of λ^2 we get the differential equation

$$s(2+s)^3V_1'(s) + (2+s)^2(s-2)V_1(s) + 4s^2 = 0$$

which can be solved using the boundary conditions (39):

$$V_1(s) = -\frac{4s \ln\left(\frac{s+2}{2}\right)}{(s+2)^2}.$$

Thus we get

$$P_1(x) = \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_1(s)\} \right] = 16e^{-2x} \left[-x^2 \{g(x) - 1\} + g(x)x - \frac{1}{4} \right]$$

where $g(x) = \ln(2x) + \gamma$. We compare the perturbatively obtained distributions (solid lines) with 1st and 2nd order correction, with $P(x)$ obtained from Monte-Carlo simulations of system size $L = 100$, for $\lambda = 0.1$ in Fig. 7(a) and (b) respectively.

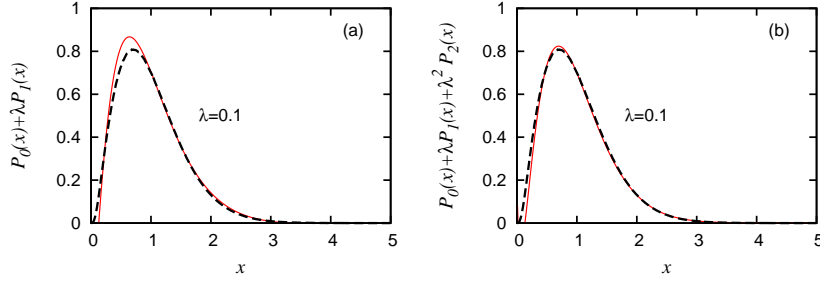


Figure 7. The distribution functions obtained perturbatively (solid lines) with 1st and 2nd order correction are compared with the results obtained from Monte-Carlo simulations (dashed lines) of SSCM with parallel dynamics of system size $L = 100$ for $\lambda = 0.1$.

2nd order perturbation in λ :

A differential equation in $V_2(s)$ can be obtained from coefficient of λ^2 in Eq. (60),

$$s(2+s)^4 V_2'(s) + (2+s)^3(s-2)V_2(s) - 16s \ln\left(\frac{s+2}{2}\right) \left\{ s^2 + s \ln\left(\frac{s+2}{2}\right) + 2 \right\} - s^2 \{s(3s+4) + 24\} = 0.$$

Using the boundary conditions (39) one can obtain the solution,

$$V_2(s) = \frac{s}{3\tilde{s}^3} \left[9s^2 + (36 - 4\pi^2)s - 8\pi^2 + 24 \ln\left(\frac{\tilde{s}}{2}\right) \left\{ \tilde{s} \ln\left(-\frac{s}{\tilde{s}}\right) + 2 - \tilde{s} \right\} + 12(s-2) \ln^2\left(\frac{\tilde{s}}{2}\right) + 24\tilde{s} Li_2\left(\frac{\tilde{s}}{2}\right) \right]$$

where, $\tilde{s} = s + 2$, $Li_2(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^2}$, is the Polylog function [20, 21].

Thus we obtain

$$P_2(x) = \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_2(s)\} \right] = 8e^{-2x} \left[4x^3 \left\{ g^2(x) - 2g(x) + 4h(2x) + \frac{11}{4} - \frac{\pi^2}{6} \right\} + 8x^2 \left\{ -g^2(x) + \frac{1}{2}g(x) \{2 - Ei(2x)\} - 2h(2x) + \frac{1}{2}f(2x) - \frac{7}{4} + \frac{\pi^2}{6} \right\} + 2x \left\{ g^2(x) + 2g(x) \{1 + Ei(2x)\} - \frac{\pi^2}{6} - 2f(2x) + e^{2x} Ei(-2x) \right\} - \{g(x) + 1\} - e^{2x} Ei(-2x) \right] + 6\delta(x)$$

where, $g(x) = \ln(2x) + \gamma$, $Ei(x) = -\int_{-x}^{\infty} dt \frac{e^{-t}}{t}$, is the exponential integral [20], $h(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!(m+1)^3}$ and $f(x) = \sum_{m=0}^{\infty} \frac{1}{m} [\gamma + \psi^{(0)}(m)] \frac{x^m}{m!}$ with $\psi^{(0)}(m) = \frac{\Gamma'(m)}{\Gamma(m)}$. In the expression for $P_2(x)$ the term $6\delta(x)$ arises in order to satisfy the normalization condition (4). The mass distribution function up to 2nd order correction

$P(x) = P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x)$ is compared with $P(x)$ obtained from Monte-Carlo simulations of a system of size $L = 100$, in Fig. 7(b). The figure shows that compared to the 1st order correction, the perturbative result up to the 2nd order correction is closer to $P(x)$ obtained from Monte-Carlo simulations. For better accuracy one has to go to higher orders.

In the following section we study SSCM with random sequential dynamics.

4.2. Random sequential update :

In this subsection we study the *symmetric sticky chipping model* using random sequential update. The dynamics is as follows. In each step choose *one* site i randomly. Of the mass x_i , the amount λx_i sticks to the site, $(1 - \lambda)r_i x_i$ is transported to the right neighbor $i + 1$ and $(1 - \lambda)(1 - r_i)x_i$ is transported to the left neighbor $i - 1$. Assuming that the steady state has a product measure, the moments $\langle x^n \rangle$ can be expressed as

$$\begin{aligned} \langle x^n \rangle &= \frac{1}{3} \langle (\lambda x_i)^n \rangle + \frac{1}{3} (1 - \lambda)^n \langle (x_i + r_{i-1} x_{i-1})^n + \{x_i + (1 - r_{i+1}) x_{i+1}\}^n \rangle \\ &= \frac{1}{3} \langle (\lambda x_i)^n \rangle + \frac{2}{3} (1 - \lambda)^n \langle (x_i + r_{i-1} x_{i-1})^n \rangle \end{aligned} \quad (61)$$

i.e., on the average one third of the time λx_i sticks to the site i , one third of the time $(1 - \lambda)r_{i-1} x_{i-1}$ is transported to site i from site $i - 1$ and one third of the time $(1 - \lambda)(1 - r_{i+1}) x_{i+1}$ is transported to site i from site $i + 1$.

Now, consider the case $\lambda = 0$. Then, Eq. (61) takes the form

$$\langle x^n \rangle = \frac{2}{3} \langle (x_i + r_{i-1} x_{i-1})^n \rangle.$$

Starting from this equation the higher moments can be expressed in terms of the lower moments in the form of recursive relation.

$$\frac{(n-1)}{2} \frac{\langle x^n \rangle}{(n+1)!} = \sum_{k=1}^{n-1} (n-k+1) \frac{\langle x^{n-k} \rangle}{(n-k+1)!} \frac{\langle x^k \rangle}{(k+1)!}$$

or,

$$(n-1)A_0^{(n)} = 2 \sum_{k=1}^{n-1} (n-k+1)A_0^{(k)}A_0^{(n-k)} \quad (62)$$

where, $A_0^{(n)} = \frac{\langle x^n \rangle}{(n+1)!}$. The generating function for $A_0^{(n)}$ is $V_0(s) = \sum_{n=0}^{\infty} (-s)^n A_0^{(n)}$. Multiplying both sides of Eq. (62) by $(-s)^n$ and taking sum over n we obtain a relation in terms of the generating function $V_0(s)$:

$$V_0'(s) = \frac{1 - 3V_0(s) + 2V_0^2(s)}{s\{3 - 2V_0(s)\}}.$$

This can be solved using the boundary condition Eq. (39) and we get

$$V_0(s) = \frac{2s - 1 + \sqrt{4s + 1}}{4s}$$

which results in

$$P_0(x) = \mathcal{L}^{-1} \left[\frac{d}{ds} \{sV_0(s)\} \right] = \frac{1}{\sqrt{4\pi}} x^{-1/2} e^{-x/4} + \delta(x). \quad (63)$$

In the above expression the term $\delta(x)$ is needed to ensure the normalization condition (3).

We compare the perturbative result with the Monte-Carlo simulations of a system of size $L = 100$ in Fig. 8.

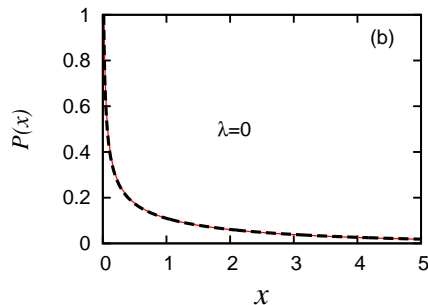


Figure 8. Comparison of Eq. (63) (solid line) with $P(x)$ obtained from Monte-Carlo simulations (dashed line) of SSCM with random sequential dynamics for $\lambda = 0$ of a system of size $L = 100$.

The Fig. 8 shows that Eq. (63) does not agree with the result obtained from Monte-Carlo simulations. Thus, in this case the assumption that the steady state has a product measure, fails badly. We do not proceed further in this case.

5. Summary and conclusion :

In this article we have studied conserved mass transfer processes using different models in one dimension with the mass treated as a continuous variable. In these chipping models at each step of update the mass at a site is distributed randomly among the site and its neighbors. We introduced a stickiness parameter λ ($0 \leq \lambda < 1$) denoting the fraction of mass that sticks to the site. Thus, a fixed fraction $(1 - \lambda)$ is transported to the neighbors. We have studied the models with asymmetric and symmetric dynamics with both parallel and random sequential update.

Finding the stationary state distribution analytically, is not always possible. We introduce a perturbative approach to obtain the approximate distribution function assuming that the steady state has a product measure. We expand the distribution function about $\lambda = 0$ as $P(x) = \sum_{k=0}^{\infty} \lambda^k P_k(x)$, and obtain the distributions up to 2nd order correction, $P(x) = P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x)$. This gives the correct values of all the moments ($\langle x^n \rangle$) up to the order λ^2 . We compare our perturbative results with that obtained from Monte-Carlo simulations, which shows that the perturbative results agrees quite well with that obtained from Monte-Carlo simulation. This indicate that product measure assumptions for the steady state holds well. For the special case of $\lambda = 0$ in all the models studied in this article and for the *asymmetric sticky chipping model* with random sequential update at $\lambda = 1/2$, we have obtained the closed form expression of

the distribution function. However, Fig. (8) clearly indicate that the assumption that the steady state has a product measure, fails badly in case of *symmetric sticky chipping model* with random sequential update.

From the perturbative series of $P(x)$ in λ up to 2^{nd} order it is not always possible to find a general expression for $P_k(x)$. In the sticky chipping models studied in this article, the mass distribution can be expressed generically in the form $P_k(x) = \sum_{l,m} G_k^{(lm)} x^l \ln^m(x)$ where $G_k^{(lm)}$ are constant coefficients. Usually the sum runs over finite number of terms (here $l \leq k+1$ and $m \leq k$). Thus for small k , one can always obtain $G_k^{(lm)}$ explicitly. Getting a generic expression, however, is not always possible. The existences of the logarithmic term in the series expansion, illustrates the complexity and “slow” convergence of $P(x)$ to a limiting function.

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